ELASTOPLASTIC PLANE CONTACT PROBLEM OF THE NONLINEAR THEORY OF ELASTICITY FOR A MATERIAL OF THE HARMONIC TYPE

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An elastoplastic plane contact problem [1] of the nonlinear theory of elasticity is studied for a half-plane of a material of the harmonic type [2] under simple loading conditions. Friction is absent in the contact region. We present the solution of an elastoplastic problem on the tension of an infinite plate of the material when it is weakened by two semiinfinite slits located along the real axis [3]. The exact solution of both of these problems is obtained.

1. Let a nonlinearly elastic half-plane S occupy the lower part of the plane of the complex variable z = x + iy. Furthermore, let a rigid die be applied along the line L' = [−b; b] of the boundary of S, henceforth designated as L. The die is applied symmetrically relative to the coordinate origin. The die is acted upon by a concentrated force $(0, N_0)$ directed vertically downward along the axis 0y, where $N_0$ is an assigned constant. The remaining part of the boundary (L" = ]−∞; −b[ ∪ ]b; ∞[) is free of external forces. There are no stresses or rotation at infinity.

Given a certain value of the external load on the contact region under the die, a plastic region of zero thickness is formed near the greatest concentration of contact stresses. This region is localized on the boundary of the half-plane under the die in the form of the zone Γ'. It is unknown beforehand and is subject to determination during the solution of the problem. We then introduce the notation Γ = L'\Γ'. We assume that the familiar Tresca–Saint Venant yield condition [4] is satisfied in the plastic region. We further assume that the die can only move translationally.

Below, we examine the case of small (but not necessarily infinitesimal) elastoplastic strains. Here, the boundary conditions (physical formulation) of the problem can be considered correct [3, 4].

Using the well-known properties of the slip line, we write the boundary conditions of the problem in the form [5]

$$X_y = 0 \text{ on } L,\ v = f(x) + \text{const} \text{ on } \Gamma; \tag{1.1}$$

$$Y_y = \sigma_s \text{ on } \Gamma',\ Y_y = 0 \quad \text{on } L'', \tag{1.2}$$

where $\sigma_s$ is the yield point in compression; y = f(x) is a real-valued function on Γ which characterizes the form of the base of the die [it is assumed that f'(x) ∈ H(Γ)].

We will solve the problem by using complex representations of the stresses, strain, and displacements for a nonlinearly-elastic material of the harmonic type in terms of two analytic functions φ(z) and ψ(z) [6] (z* = z + u + iv) that are analytic in the physical region S:

$$X_x + Y_y + 4\mu = \frac{\lambda+2\mu}{\sqrt{J}} q\Omega(q),\quad Y_y - X_x - 2iX_y = -\frac{4(\lambda+2\mu)}{\sqrt{J}}\frac{\Omega(q)}{q}\frac{\partial z^*}{\partial z}\frac{\partial z^*}{\partial \bar z}; \tag{1.3}$$

$$\frac{\partial z^*}{\partial z} = \frac{\mu}{\lambda+2\mu}\varphi'^2(z) + \frac{\lambda+\mu}{\lambda+2\mu}\frac{\varphi'(z)}{\overline{\varphi'(z)}},\quad \frac{\partial z^*}{\partial \bar z} = -\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}} - \overline{\psi'(z)}\right]; \tag{1.4}$$

$$u + iv = \frac{\mu}{\lambda+2\mu}\int\varphi'^2(z)\,dz + \frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)}{\overline{\varphi'(z)}} + \overline{\psi(z)}\right] + \text{const}; \tag{1.5}$$

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$$\sqrt{J} = \frac{\partial z^*}{\partial z}\frac{\partial \bar{z}^*}{\partial \bar{z}} - \frac{\partial z^*}{\partial \bar{z}}\frac{\partial \bar{z}^*}{\partial z}, \quad q = 2\left|\frac{\partial z^*}{\partial z}\right|, \; \Omega(q) = q - \frac{2(\lambda+\mu)}{\lambda+2\mu}, \tag{1.6}$$

$$z^* = z + u + iv.$$

It was shown in [6] that for large $|z|$ there exist the following representations (in the absence of stresses and rotation at infinity):

$$\begin{aligned} \varphi(z) &= -\frac{(\lambda+2\mu)(X+iY)}{4\pi\mu(\lambda+\mu)}\ln z + z + \varphi_0(z) + \text{const}, \\ \psi(z) &= \frac{(\lambda+2\mu)(X-iY)}{2\pi\mu(\lambda+\mu)}\left[\frac{1}{2\varphi'(z)} - 1\right]\ln z + \psi_0(z) + \text{const}. \end{aligned} \tag{1.7}$$

Here (X, Y) is the principal vector of all of the external forces; $\lambda$ and $\mu$ are the Lamé constants; $\varphi_0(z)$ and $\psi_0(z)$ are functions which are holomorphic in S and which have the order o(1) at large $|z|$. Also,

$$\varphi'(z) \neq 0 \text{ in } S+L. \tag{1.8}$$

On the basis of the first equation of (1.1), we can use (1.3), (1.4), (1.6), and (1.7) to arrive at the relation

$$\overline{\varphi(x)}\,\varphi''(x) - \varphi'^2(x)\,\psi'(x) = 0 \quad \text{on } L. \tag{1.9}$$

With the use of (1.9), comparison of (1.3) and (1.4) produces a formula which is important for further examination of the problem

$$Y_y = N(x) = \frac{2\mu(\lambda+\mu)\left[|\varphi'^2(x)| - 1\right]}{\lambda+\mu+\mu|\varphi'^2(x)|} \quad \text{on } L. \tag{1.10}$$

Proceeding on the basis of (1.7), it can readily be seen from (1.10) that

$$\varphi'(z) = \exp\left(-\frac{1}{\pi i}\int\limits_L \frac{F(x)\,dx}{x-z}\right) \text{ with } z \in S; \tag{1.11}$$

$$F(x) = \frac{1}{2}\ln\left[\frac{\lambda+\mu}{\mu}\,\frac{2\mu+N(x)}{2(\lambda+\mu)-N(x)}\right]. \tag{1.12}$$

From here, in accordance with (1.2), we have

$$\varphi'(z) = \exp\left\{-\frac{F_s}{\pi i}[\ln(x-z)]_{\Gamma'} - \frac{1}{\pi i}\int\limits_\Gamma \frac{F(x)\,dx}{x-z}\right\}. \tag{1.13}$$

Now let us examine boundary value (1.5) on L and let us differentiate the resulting equation with respect to x. Then, with allowance for (1.9),

$$u_x' + iv_x' = \varphi'^2(x)\left[\frac{\mu}{\lambda+2\mu} + \frac{\lambda+\mu}{\lambda+2\mu}\,\frac{1}{|\varphi'^2(x)|}\right] - 1 \text{ on } L.$$

From here, we obtain the following on the basis of the second equation of (1.1):

$$\left[\mu + \frac{\lambda+\mu}{|\varphi'^2(x)|}\right]\operatorname{Im}\varphi'^2(x) = (\lambda+2\mu)f'(x) \text{ on } \Gamma.$$

Now we calculate the boundary value $\varphi'(z)$, given by Eq. (1.13), when z approaches the point x of the contour $\Gamma$ while remaining inside S. Here we introduce the notation $[\ln(x - x_0)]_{\Gamma'} = A(x_0)$. Then using the well-known Sokhostkii–Plemelj formula, we obtain the following on $\Gamma$:

$$[\lambda+\mu+\mu\exp(2F(x_0))]\sin\left[\frac{2F_s}{\pi}A(x_0) + \frac{2}{\pi}\int\limits_\Gamma \frac{F(x)\,dx}{x-x_0}\right] = (\lambda+2\mu)f'(x_0), \tag{1.14}$$

where F(x) is determined in accordance with (1.12), while $F_s$ is constant:

$$F_s = \frac{1}{2}\ln\left(\frac{\lambda+\mu}{\mu}\,\frac{2\mu+\sigma_s}{2(\lambda+\mu)-\sigma_s}\right). \tag{1.15}$$

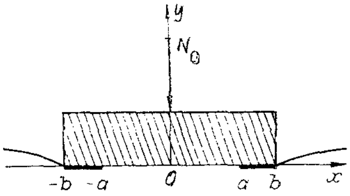


Fig. 1

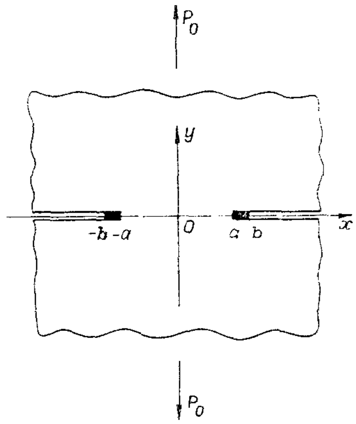


Fig. 2

Equation (1.14) is a nonlinear functional equation for determining the function F(x) on Γ. Also, the integration line is unknown. Thus, this equation does not belong to one of the familiar (investigated) classes of nonlinear equations.

Despite these difficulties, an exact solution for the problem and a unique solution for the given class can be obtained in one special case of practical importance.

2. We will examine a rigid die having a rectangular horizontal base, i.e., when f(x) = const. We further put $\Gamma' = [-b; -a] \cup [a; b]$ and, consequently, $\Gamma = ]-a; a[$. This means that the plastic region consists of two rectilinear segments $[-b; -a]$, $[a; b]$ located on the contact line next to corner points −b and b (Fig. 1).

In the present case, $A(x_0) = \ln \frac{(x_0+a)(x_0-b)}{(x_0-a)(x_0+b)}$. Thus, Eq. (1.18) takes the form

$$\int_{-a}^{a} \frac{F(x)\,dx}{x-x_0} = -F_s \ln \frac{(x_0+a)(x_0-b)}{(x_0-a)(x_0+b)} + C,\ x,\ x_0 \in ]-a; a[ \tag{2.1}$$

(C is an arbitrary constant). Considering that F(−x) = F(x) and assuming in (2.1) that $x_0$ = 0, we obtain C = 0 and, thus,

$$\int_{-a}^{a} \frac{F(x)\,dx}{x-x_0} = -F_s \ln \frac{(x_0+a)(x_0-b)}{(x_0-a)(x_0+b)}.$$

As a result, we have found a singular characteristic integral equation to determine the function F(x) on the interval $]-a; a[$. As is known, the solution of this equation of the class $h(-a; a)$ has the form [7]

$$F(x_0) = \frac{F_s\sqrt{a^2-x_0^2}}{\pi^2} \int_{-a}^{a} \frac{\ln[(x+a)(x-b) \mid (x-a)(x+b)]}{\sqrt{a^2-x^2}} \frac{dx}{x-x_0} \tag{2.2}$$

when the following solvability condition is satisfied:

$$\int_{-a}^{a} \frac{\ln[(x+a)(x-b) \mid (x-a)(x+b)]\,dx}{\sqrt{a^2-x^2}} = 0.$$

However, since the integrand is odd, this condition is satisfied automatically.

Let us calculate the singular integral in the right side of (2.2). After reductions, we obtain

$$F(x) = F_s\left[1 + \frac{1}{\pi}\operatorname{arctg}\sqrt{\frac{a^2-x^2}{b^2-a^2}}\right] \tag{2.3}$$

($-a < x < a$). This formula contains the unknown constant $a$. Let us determine it. First we insert (2.3) into the right side of (1.11). Then, after performing the necessary calculations, we find

$$\varphi'(z) = \exp\left[\frac{F_s}{\pi i}\ln\frac{(z+b)\sqrt{(b^2-a^2)(z^2-a^2)}+bz-a^2}{(z-b)\sqrt{(b^2-a^2)(z^2-a^2)}+bz+a^2}\right], \tag{2.4}$$

where the branch of the function $\sqrt{z^2-a^2}$ is fixed by the condition $\lim_{z\to\infty}\frac{\sqrt{z^2-a^2}}{z}=1$, and the constant $F_s$ is determined by Eq. (1.15).

We calculate the asymptote with $z^{-1}$ in the right side of (2.4) and we compare the resulting expression with the corresponding expression from Eqs. (1.7). After performing the necessary calculations, we obtain

$$a = b\sqrt{1-\frac{(\lambda+2\mu)^2N_0^2}{16\mu^2(\lambda+\mu)^2b^2\left[\ln\left(\frac{\lambda+\mu}{\mu}\,\frac{2\mu+\sigma_s}{2(\lambda+\mu)-\sigma_s}\right)\right]^2}}. \tag{2.5}$$

In accordance with the linear classical theory, this formula appears as [2]

$$a = b\sqrt{1-\frac{N_0^2}{4b^2\sigma_s^2}}.$$

Let us now return to Eq. (2.3). Allowing for (1.12), we find from (2.3) that

$$N(x) = \frac{2\mu\left[\left(\frac{\lambda+\mu}{\mu}\,\frac{2\mu+\sigma_s}{2(\lambda+\mu)-\sigma_s}\right)^{1+\frac{2}{\pi}\operatorname{arctg}\sqrt{\frac{a^2-x^2}{b^2-a^2}}}-1\right]}{1+\frac{\mu}{\lambda+\mu}\left(\frac{\lambda+\mu}{\mu}\,\frac{2\mu+\sigma_s}{2(\lambda+\mu)-\sigma_s}\right)^{1+\frac{2}{\pi}\operatorname{arctg}\sqrt{\frac{a^2-x^2}{b^2-a^2}}}} \tag{2.6}$$

on $[-a; a]$. After we determine $\varphi(z)$, we can find the other sought function $\psi(z)$ from condition (1.9) by an established method. The field of the elastic elements of the region in question is determined from (1.3)-(1.6) by numerical operations.

3. Let us examine the following problem. Let an infinite plane $z = x + iy$ of a harmonic material be weakened by two semiinfinite straight slits located along the real axis. The edges of the slits are free of external loads, while only the concentrated force (0, $N_0$) acts at infinity. The concentrated force is applied at an infinitely distant point and is directed along the Oy axis (a force of the same magnitude, balanced by the first force, acts at the point $z = -\infty$). Now the plastic region consists of two segments located along the line of the slit on its extension (Fig. 2).

If we imaginarily remove the top half-plane, then its effect on the lower part S is identical to the action of a rigid die with a straight horizontal base on the line between the slits (cracks). However, now the die must be acted upon by a force that balances the force acting on the bottom half-plane. If we introduce the notation $L = ]-\infty; \infty[$, $L' = [-b; b]$, $\Gamma' = [-b; -a] \cup [a; b]$, $\Gamma = L'\backslash\Gamma'$, and $L'' = L\backslash L'$, then the boundary conditions of the problem have the form

$$X_y = 0 \text{ on } L,\ v' = 0 \text{ on } \Gamma,\ Y_y = \sigma_s \text{ on } \Gamma',\ Y_y = 0 \text{ on } L'',$$

i.e., they coincide with conditions (1.1) and (1.2) only if $f(x) = \text{const}$ in the latter. Thus, the complete analogy is made evident. As a result, the solution of the problem is given by Eqs. (2.4)-(2.6). In these formulas, $N_0$ is replaced by $N_0 = P_0$. This solution is shown below written in a somewhat different form. In particular, the complex potential

$$\varphi'(z) = \exp\left[-\frac{F_s}{\pi i}\ln\frac{\sqrt{z^2-a^2}-\sqrt{b^2-a^2}}{\sqrt{z^2-a^2}+\sqrt{b^2-a^2}}\right],$$

while we obtain the following expression for the normal stress on the segment $[-a; a]$ between the slits

$$N(x) = \frac{2\mu\left[\exp 2F_s\left(1+\frac{1}{\pi}\operatorname{arctg}\frac{2\sqrt{(b^2-a^2)(a^2-x^2)}}{b^2-2a^2+x^2}\right)-1\right]}{1+\frac{\mu}{\lambda+\mu}\exp 2F_s\left(1+\frac{1}{\pi}\operatorname{arctg}\frac{2\sqrt{(b^2-a^2)(a^2-x^2)}}{b^2-2a^2+x^2}\right)}.$$

The linear dimension of the plastic region is determined by the formula

$$a = b\left[1-\frac{(1-\nu^2)^2P_0^2}{4E^2b^2F_s^2}\right]^{1/2},$$

where $\nu$ is the Poisson ratio; E is the Young modulus; the constant $F_s$ is found from (1.15).

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# ALGORITHM FOR STUDYING THE NONLINEAR DEFORMATION AND STABILITY OF CIRCULAR CYLINDRICAL SHELLS WITH INITIAL SHAPE FLAWS


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Axisymmetric deflections have been examined in most of the well-known solutions of problems concerning the stability of shells with initial deflections. Some of the studies have examined the effect of nonaxisymmetric deflections. The solutions have been obtained either in a classical formulation, without allowance for the moments of the initial stress state, or in a formulation which presumed the development of initial deflections, without restructuring, during nonlinear deformation under axisymmetric loads.

Below we obtain a fairly general solution to the problem, without restrictions on the load or the form of the initial and bifurcative deflections. We use the method of finite elements in displacements. The finite elements are chosen in the form of rectangles of natural curvature having form functions which consider their displacement as rigid bodies.

We will examine a circular cylindrical shell of the length L, radius R, and thickness h. The initial shape flaws are given either by the series $w^0 = \sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}\cos i\varphi\cos j\pi x/L$, or by a two-dimensional set of nodal values of the initial deflection and its derivatives $\overline{w^0} =$


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